

STOCHASTIC HOMOGENIZATION OF NONCONVEX HAMILTON-JACOBI EQUATIONS IN ONE SPACE DIMENSION

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ABSTRACT. We prove stochastic homogenization for a general class of coercive, nonconvex Hamilton-Jacobi equations in one space dimension. Some properties of the effective Hamiltonian arising in the nonconvex case are also discussed.

1. INTRODUCTION

1.1. Motivation and overview. We study the coercive Hamilton-Jacobi equation

$$(1.1) \quad u_t^\varepsilon + H(Du^\varepsilon) + V\left(\frac{x}{\varepsilon}\right) = 0 \quad \text{in } \mathbb{R} \times (0, \infty).$$

The Hamiltonian $H : \mathbb{R} \rightarrow \mathbb{R}$ is a deterministic function which satisfies $H(p) \rightarrow +\infty$ as $|p| \rightarrow +\infty$. In particular, we do not assume H is convex. The potential V is a bounded, stationary random field sampled by an ergodic probability measure. We prove that, in the limit as the length scale $\varepsilon > 0$ of the correlations tends to zero, the solution u^ε of (1.1), subject to an appropriate initial condition, converges to the solution u of the effective, deterministic equation

$$(1.2) \quad u_t + \overline{H}(Du) = 0 \quad \text{in } \mathbb{R} \times (0, \infty).$$

The random homogenization of Hamilton-Jacobi equations has received much attention in the last fifteen years. The first results were due to Rezakhanlou and Tarver [14] and Souganidis [16], who independently proved qualitative results for general convex, first-order Hamilton-Jacobi equations in stationary-ergodic setting. Later, these results were extended to the viscous case by Kosygina, Rezakhanlou and Varadhan [8] and, independently, by Lions and Souganidis [11] as well as to equations with time-dependent coefficients by Kosygina and Varadhan [9] and Schwab [15]. New proofs of these results based on the notion of intrinsic distance functions appeared later in Armstrong and Souganidis [3] for the first-order case and in Armstrong and Tran [4] in the viscous case. The latter allowed for quantitative results, which appeared in Armstrong, Cardaliaguet and Souganidis [2] (see also Matic and Nolen [12]) and Armstrong and Cardaliaguet [1].

There are however few such results for equations which are not convex (or, at least not quasi-convex, see [3, 6]) in the gradient variable— a fact which has been highlighted as one of the prominent open problems in the field. Essentially the only

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previous result for a genuinely non-convex equation is due to the authors [5]. In that paper, we proved that the equation

$$u_t^\varepsilon + (|Du^\varepsilon|^2 - 1)^2 + V\left(\frac{x}{\varepsilon}\right) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty)$$

homogenizes for stationary-ergodic potentials in all space dimensions $d \geq 1$. Using some ideas from [5], in this paper we prove, in $d = 1$, that the special nonconvex gradient profile in the latter equation may be replaced by a general coercive function. Although our arguments are confined to one space dimension, this is the first stochastic homogenization result for a general class of nonconvex Hamilton-Jacobi equations.

As we will see, the main difficulty is to analyze the precise shock structure of solutions of (1.1), in particular with the way the potential interacts “non-locally” with the bumps in the graph of the Hamiltonian H . We eventually argue by induction, removing some bumps at a time until we are left with a quasi-convex equation or the situation in Section 3 (the oscillation of V is larger than the global oscillation of all such bumps) where homogenization result is obtained straightly.

1.2. Precise statement of the main result. The random potential is modeled by a probability measure on the set of all potentials. More precisely, let

$$\Omega := \text{BUC}(\mathbb{R})$$

be the space of real-valued, bounded and uniformly continuous functions on \mathbb{R} . We define \mathcal{F} to be the σ -algebra on Ω generated by pointwise evaluations:

$$\mathcal{F} := \sigma\text{-algebra generated by the family of maps } \{V \mapsto V(x) : x \in \mathbb{R}\}.$$

The translation group action of \mathbb{R} on Ω is denoted by $\{T_y\}_{y \in \mathbb{R}}$ where $T_y : \Omega \rightarrow \Omega$ is defined by

$$(T_y V)(x) := V(x + y).$$

We consider a probability measure \mathbb{P} on (Ω, \mathcal{F}) satisfying the followings: there exists $\overline{m} > 0$ such that

$$(1.3) \quad \mathbb{P} \left[\text{ess sup}_{x \in \mathbb{R}} V(x) = 0 \right] = 1 \quad \text{and} \quad \mathbb{P} \left[\text{ess sup}_{x \in \mathbb{R}} (-V(x)) = \overline{m} \right] = 1$$

for every $E \in \mathcal{F}$ and $y \in \mathbb{R}$,

$$(1.4) \quad \mathbb{P}[E] = \mathbb{P}[T_y E] \quad (\text{stationarity})$$

and

$$(1.5) \quad \mathbb{P} \left[\bigcap_{z \in \mathbb{R}} T_z E \right] \in \{0, 1\} \quad (\text{ergodicity}).$$

Assume that $H \in C(\mathbb{R})$ and

$$(1.6) \quad \lim_{|p| \rightarrow \infty} H(p) = +\infty.$$

Notice that, by ergodicity, there is no loss in generality in (1.3) compared to the assumption that $\mathbb{P}[\text{ess sup}_{x \in \mathbb{R}^d} |V(x)| < \infty] = 1$.

We now present the main result. Throughout, all differential equations and inequalities in this paper are to be interpreted in the viscosity sense (see [7]). Recall

that, for each $\varepsilon > 0$ and $g \in \text{BUC}(\mathbb{R})$, there exists a unique solution $u^\varepsilon(\cdot, g) \in C(\mathbb{R} \times [0, \infty))$ of (1.1) in $\mathbb{R} \times (0, \infty)$, subject to the initial condition $u^\varepsilon(x, 0, g) = g(x)$.

Theorem 1.1. *Assume (1.3)–(1.6) hold. Then there exists $\bar{H} \in C(\mathbb{R})$ satisfying*

$$(1.7) \quad \bar{H}(p) \rightarrow +\infty \quad \text{as } |p| \rightarrow \infty$$

such that, if we denote, for each $g \in \text{BUC}(\mathbb{R})$, the unique solution of (1.2) subject to the initial condition $u(x, 0) = g(x)$ by $u(x, t, g)$, then

$$\mathbb{P} \left[\forall g \in \text{BUC}(\mathbb{R}), \forall k > 0, \limsup_{\varepsilon \rightarrow 0} \sup_{(x,t) \in B_k \times [0,k]} |u^\varepsilon(x, t, g) - u(x, t, g)| = 0 \right] = 1.$$

We highlight two key properties of \bar{H} which play significant roles in our proof.

- *Quasi-convexification of the effective Hamiltonian.* As we will show, the effective Hamiltonian $\bar{H}(p)$ becomes quasi-convex when \bar{m} is large (see Theorem 3.11). Thus when the oscillation of the potential is large, we may expect the effective Hamiltonian to be “less non-convex” than H . Similar facts have also been noticed in [13] and [5].
- *Existence and nonexistence of sublinear correctors.* In the random setting, a simple example due to Lions and Souganidis [10] shows that the cell problem might not have sublinear solutions. As we will see, our proof actually demonstrates that, in $d = 1$, sublinear correctors exist away from those flat pieces of \bar{H} where \bar{H} attains local extreme values.

2. PRELIMINARIES

Definition 2.1. We say that the pair (H, V) is *regularly homogenizable* (with respect to \mathbb{P}) at $p \in \mathbb{R}$ if (H, V) satisfies (1.3)–(1.6), and there exists $\bar{H}(p) \in \mathbb{R}$ such that for any $R > 0$,

$$(2.1) \quad \mathbb{P} \left[\limsup_{\lambda \rightarrow 0} \max_{|y| \leq R/\lambda} |\lambda v_\lambda(y, p) + \bar{H}(p)| = 0 \right] = 1,$$

where $v_\lambda(\cdot, p)$ is the unique continuous bounded viscosity solution to

$$(2.2) \quad \lambda v_\lambda + H(p + v'_\lambda) + V(y) = 0 \quad \text{in } \mathbb{R}.$$

(H, V) is called *regularly homogenizable* if (H, V) is regularly homogenizable at p for every $p \in \mathbb{R}$.

The merit of this definition is that the conclusion of Theorem 1.1 holds if (H, V) is regularly homogenizable (see for example [2, Lemma 7.1]). Moreover, in view of [3, Lemma 5.1], the condition (2.1) is equivalent to the following seemingly weaker convergence assertion:

$$(2.3) \quad \mathbb{P} \left[\lim_{\lambda \rightarrow 0} |\lambda v_\lambda(0, p) + \bar{H}(p)| = 0 \right] = 1.$$

In order to obtain Theorem 1.1, it is enough to prove the following statement.

Theorem 2.2. *Assume (1.3)–(1.6) hold. Then (H, V) is regularly homogenizable.*

We next notice that, by comparison principle, the property of being regularly homogenizable is stable under the supremum norm. The proof is easy and thus we omit it.

Lemma 2.3. *Assume that (H_n, V_n) is regularly homogenizable at $p \in \mathbb{R}$ for each $n \in \mathbb{N}$, and there exists (H, V) such that*

$$\lim_{n \rightarrow \infty} (\|H_n - H\|_{L^\infty(\mathbb{R})} + \|V_n - V\|_{L^\infty(\mathbb{R} \times \Omega)}) = 0.$$

Then (H, V) is also regularly homogenizable at $p \in \mathbb{R}$ and

$$\overline{H}(p) = \lim_{n \rightarrow \infty} \overline{H}_n(p).$$

By Lemma 2.3 and Lemma A.5 in the appendix, we may assume in addition to (1.3)–(1.6) the following assumptions throughout the paper

(H1) $H : \mathbb{R} \rightarrow [0, \infty)$ is Lipschitz continuous, $\min_{\mathbb{R}} H = H(0) = 0$ and

$$\lim_{|p| \rightarrow \infty} H(p) = +\infty.$$

(H2) There exist $L \geq 0$ and $p_1 > p_2 > \dots > p_{2L} > p_{2L+1} = 0$ such that

- (i) H is strictly increasing in $[p_1, \infty)$ and $[p_{2k+1}, p_{2k}]$ for $1 \leq k \leq L$,
- (ii) H is strictly decreasing in $[p_{2k}, p_{2k-1}]$ for $1 \leq k \leq L$.
- (iii) $H(p_1), H(p_2), \dots, H(p_{2L+1})$ are distinct positive numbers.

(H3) There exist $\tilde{L} \geq 0$ and $\tilde{p}_1 < \tilde{p}_2 < \dots < \tilde{p}_{2\tilde{L}+1} = 0$ such that

- (i) H is strictly decreasing in $(-\infty, \tilde{p}_1]$ and $[\tilde{p}_{2k}, \tilde{p}_{2k+1}]$ for $1 \leq k \leq \tilde{L}$,
- (ii) H is strictly increasing in $[\tilde{p}_{2k-1}, \tilde{p}_{2k}]$ for $1 \leq k \leq \tilde{L}$.
- (iii) $H(\tilde{p}_1), H(\tilde{p}_2), \dots, H(\tilde{p}_{2\tilde{L}+1})$ are distinct positive numbers.

(H4) $V \in C^\infty(\mathbb{R})$ and each level set of V has no cluster points, that is, there does not exist any $y \in \mathbb{R}$ such that $V^{(k)}(y) = 0$ for all $k \in \mathbb{N}$.

Set

$$m_i := H(p_{2i-1}) \quad \text{and} \quad M_i := H(p_{2i}) \quad \text{for } i = 1, \dots, L.$$

We denote

$$\phi_1 := H|_{[p_1, \infty)}, \quad \phi_i := H|_{[p_i, p_{i-1}]} \quad \text{for } 2 \leq i \leq 2L+1,$$

and

- $\psi_1 : [m_1, \infty) \rightarrow [p_1, \infty)$ as $\psi_1 := \phi_1^{-1}$,
- $\psi_{2i} : [m_i, M_i] \rightarrow [p_{2i}, p_{2i-1}]$ as $\psi_{2i} := \phi_{2i}^{-1}$ for $1 \leq i \leq L$,
- $\psi_{2i-1} : [m_i, M_{i-1}] \rightarrow [p_{2i-1}, p_{2i-2}]$ as $\psi_{2i-1} := \phi_{2i-1}^{-1}$ for $2 \leq i \leq L+1$.

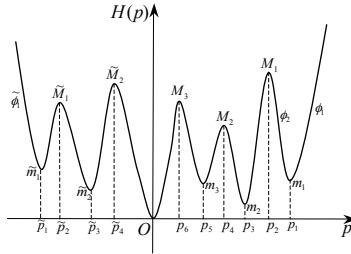


FIGURE 2.1. Graph of H with $L = 3$ and $\tilde{L} = 2$

Definition 2.4. We say that $f \in \mathcal{A}(H, V, \mu)$ for $\mu \in \mathbb{R}$ if $f \in L^\infty(\mathbb{R})$ and any $u \in C^{0,1}(\mathbb{R})$ solution to $u' = f$ is a solution to

$$(2.4) \quad H(u') + V(y) = \mu \quad \text{in } \mathbb{R}.$$

Lemma 2.5. Suppose that u and v are both viscosity solutions to

$$\lambda w + H(p + w') + V(y) = 0 \quad \text{in } B_{R/\lambda},$$

for some $R > 0$ and $\lambda \in (0, 1)$. Assume that $\lambda(|u| + |v|) \leq C$ in $B_{R/\lambda}$ and $\|H'\|_{L^\infty(\mathbb{R})} \leq C$ for some $C > 0$. Then

$$(2.5) \quad \lambda|u(y) - v(y)| \leq \frac{C}{R} (|y|^2 + 1)^{1/2} + \frac{C^2}{R} \quad \text{for } y \in B_{R/\lambda}.$$

Proof. Let $w(y) := v(y) + \frac{C}{R} (|y|^2 + 1)^{1/2} + \frac{C^2}{R\lambda}$ for $y \in B_{R/\lambda}$. It is straightforward that w is a viscosity supersolution to

$$\lambda w + H(p + w') + V(y) = 0 \quad \text{in } B_{R/\lambda},$$

and $u \leq w$ on $\partial B_{R/\lambda}$. Thus $u \leq w$ in $B_{R/\lambda}$. \square

Lemma 2.6 (Generalized mean value theorem). Suppose that $u \in C([0, 1], \mathbb{R})$ and, for some $a, b \in \mathbb{R}$,

$$u'(0^+) = \lim_{x \rightarrow 0^+} \frac{u(x) - u(0)}{x} = a \quad \text{and} \quad u'(1^-) = \lim_{x \rightarrow 1^-} \frac{u(1) - u(x)}{1 - x} = b.$$

Then: (i) If $a < b$, then for any $c \in (a, b)$, there exists $x_c \in (0, 1)$ such that $c \in D^-u(x_c)$, i.e.,

$$u(x) \geq u(x_c) + c(x - x_c) - o(|x - x_c|) \quad \text{for } x \in (0, 1).$$

(ii) If $a > b$, then for any $c \in (b, a)$, there exists $x_c \in (0, 1)$ such that $c \in D^+u(x_c)$, i.e.,

$$u(x) \leq u(x_c) + c(x - x_c) + o(|x - x_c|) \quad \text{for } x \in (0, 1).$$

Proof. It is enough to prove (i). For $c \in (a, b)$, set $w(x) := u(x) - cx$ for $x \in [0, 1]$. There exists $x_c \in [0, 1]$ such that

$$w(x_c) = \min_{x \in [0, 1]} w(x).$$

Note that $x_c \neq 0$ and $x_c \neq 1$ as $c \in (a, b)$. Thus $x_c \in (0, 1)$, which of course yields that $c \in D^-u(x_c)$. \square

3. HOMOGENIZATION IN CASE THE OSCILLATION OF V IS LARGE

In this section, we assume that $\tilde{L} = 0$, and

$$(3.1) \quad \bar{m} > \max_{1 \leq i, j \leq L} (M_i - m_j),$$

and set $m_{\min} := \min_{1 \leq i \leq L} m_i > 0$, $M_{\max} := \max_{1 \leq i \leq L} M_i > 0$, and

$$\mathcal{P} := \{\mu \geq 0 : \mu \in (m_{\min} - \bar{m}, M_{\max})\}.$$

Definition 3.1. For $\mu \in \mathcal{P}$, a collection of finite intervals $\{I_i\}_{i \in \mathbb{Z}}$ is called a (V, μ) -admissible decomposition of \mathbb{R} if

$$I_i = (a_i, a_{i+1}), \quad \lim_{i \rightarrow \pm\infty} a_i = \pm\infty, \quad \mu - V(a_i) \in \{m_j, M_j : 1 \leq j \leq L\},$$

and $\{\mu - V(y) : y \in I_i\} \cap \{m_j, M_j : 1 \leq j \leq L\} = \emptyset$.

Owing to (3.1), (H4), and Lemma A.6, $\{I_i\}$ exists and is unique up to a translation of indices in \mathbb{Z} . Furthermore, for any $i \in \mathbb{Z}$ and $y \in \mathbb{R}$,

$$T_y I_i = I_i + y.$$

Definition 3.2. For $\mu \in \mathcal{P}$, we say $f \in \mathcal{A}(H, V, \mu)$ is furthermore (I_i, V, μ) -admissible if

$$0 \leq f \leq \max\{p \geq 0 : H(p) \leq \mu + \overline{m}\},$$

$$f|_{I_i} = \psi_{j_i}(\mu - V) \quad \text{for some } j_i \in \{1, \dots, 2L+1\}, \quad \text{for all } i \in \mathbb{Z}.$$

Lemma 3.3. For each $\mu \in \mathcal{P}$, there exists an (I_i, V, μ) -admissible function f .

Proof. In view of Lemmas A.1, A.2 in the appendix, there exist a strictly increasing sequence $\{b_i\}_{i \in \mathbb{Z}}$ and a Lipschitz continuous solution u to (2.4) such that $\lim_{i \rightarrow \pm\infty} b_i = \pm\infty$, $u \in C^1((b_i, b_{i+1}))$ for all $i \in \mathbb{Z}$ and

$$u'|_{(b_i, b_{i+1})} = \psi_{k_i}(\mu - V) \quad \text{for some } k_i \in \{1, \dots, 2L+1\}.$$

By refinement, we may assume further that for each $i \in \mathbb{Z}$,

$$(b_i, b_{i+1}) \subseteq I_{l_i} \quad \text{for some } l_i \in \mathbb{Z}.$$

For each $j \in \mathbb{Z}$, set

$$s_j := \min\{k_i : (b_i, b_{i+1}) \subseteq I_j\}$$

and

$$f = \psi_{s_j}(\mu - V) \quad \text{in } I_j.$$

In light of one of the homotopy results, Lemma A.3 in the appendix, we conclude that $f \in \mathcal{A}(H, V, \mu)$ and furthermore (I_i, V, μ) -admissible. \square

We now begin the identification of the effective Hamiltonian \overline{H} . For $\mu \in [0, \infty) \setminus \mathcal{P}$, we set

$$f_\mu := \begin{cases} \psi_{2L+1}(\mu - V) & \text{if } \mu \leq m_{\min} - \overline{m}, \\ \psi_1(\mu - V) & \text{if } \mu \geq M_{\max}. \end{cases}$$

It is clear that $f_\mu \in \mathcal{A}(H, V, \mu)$ for $\mu \in [0, \infty) \setminus \mathcal{P}$.

For $\mu \in \mathcal{P}$ and $y \in \mathbb{R}$, define

$$\overline{f}_\mu(y) := \sup\{f(y) : f \text{ is } (I_i, V, \mu)\text{-admissible}\}$$

and

$$\underline{f}_\mu(y) := \inf\{f(y) : f \text{ is } (I_i, V, \mu)\text{-admissible}\}.$$

Lemma 3.4. Both \overline{f}_μ and \underline{f}_μ are stationary as well as (I_i, V, μ) -admissible.

Proof. Stationarity of \bar{f}_μ and \underline{f}_μ is straightforward. We now only check that \bar{f}_μ is (I_i, V, μ) -admissible. We notice first that for all $i \in \mathbb{Z}$,

$$\bar{f}_\mu|_{I_i} = \psi_{j_i}(\mu - V) \quad \text{for some } j_i \in \{1, \dots, 2L+1\}.$$

Thus, we only need to check that for $u \in C^{0,1}(\mathbb{R})$ such that $u' = \bar{f}_\mu$, u is a solution of (2.4) at $y = a_i$.

Pick f_1, f_2 which are (I_i, V, μ) -admissible such that

$$\bar{f}_\mu = f_1 \quad \text{in } I_{i-1} \quad \text{and} \quad \bar{f}_\mu = f_2 \quad \text{in } I_i.$$

Case 1. If

$$f_1(a_i^-) := \lim_{y \rightarrow a_i^-} f_1(y) \geq f_2(a_i^+) := \lim_{y \rightarrow a_i^+} f_2(y),$$

then it is clear that

$$D^+u(a_i) = [\bar{f}_\mu(a_i^+), \bar{f}_\mu(a_i^-)] = [f_2(a_i^+), f_1(a_i^-)] \subseteq [f_1(a_i^+), f_1(a_i^-)].$$

Case 2. If

$$f_1(a_i^-) := \lim_{y \rightarrow a_i^-} f_1(y) \leq f_2(a_i^+) := \lim_{y \rightarrow a_i^+} f_2(y),$$

then it is clear that

$$D^-u(a_i) = [\bar{f}_\mu(a_i^-), \bar{f}_\mu(a_i^+)] = [f_1(a_i^-), f_2(a_i^+)] \subseteq [f_2(a_i^-), f_2(a_i^+)].$$

The desired result follows. \square

Lemma 3.5. *For $\mu \in \mathcal{P}$ and $p \in [\mathbb{E}[\underline{f}_\mu(0)], \mathbb{E}[\bar{f}_\mu(0)]]$, there exists a stationary function f such that $f \in \mathcal{A}(H, V, \mu)$ and*

$$p = \mathbb{E}[f(0)].$$

Proof. For $i \in \mathbb{Z}$, denote

$$\bar{d}_i := \int_{a_i}^{a_{i+1}} \bar{f}_\mu(y) dy \quad \text{and} \quad \underline{d}_i = \int_{a_i}^{a_{i+1}} \underline{f}_\mu(y) dy.$$

According to (3.1) and Lemma A.6, there exists a subsequence of intervals $\{I_{k_j}\}_{j \in \mathbb{Z}}$ such that $\lim_{j \rightarrow \pm\infty} k_j = \pm\infty$ and

$$\bar{f}_\mu = \underline{f}_\mu = \psi_1(\mu - V) \quad \text{in } I_{k_j}$$

or

$$\bar{f}_\mu = \underline{f}_\mu = \psi_{2L+1}(\mu - V) \quad \text{in } I_{k_j}.$$

By annexation if necessary, we may assume that for all $i \in \mathbb{Z}$

$$\begin{cases} \bar{f}_\mu = \underline{f}_\mu & \text{in } I_{2i} \\ \bar{f}_\mu > \underline{f}_\mu & \text{in } I_{2i+1}. \end{cases}$$

For $t \in [0, 1]$ and $i \in \mathbb{Z}$, set $d_i(t) := t\bar{d}_i + (1-t)\underline{d}_i$ and

$$f_{\mu,t} := \begin{cases} \bar{f}_\mu = \underline{f}_\mu & \text{in } I_{2i}, \\ f_{d_{2i+1}(t)}(\bar{f}_\mu, \underline{f}_\mu, I_{2i+1}) & \text{in } I_{2i+1}. \end{cases}$$

By Lemma A.4, $f_{\mu,t} \in \mathcal{A}(H, V, \mu)$. The usual ergodic theorem gives

$$\mathbb{E} \left[\lim_{T \rightarrow \pm\infty} \frac{1}{T} \int_0^T f_{\mu,t}(y) dy = \mathbb{E}(f_{\mu,t}(0)) \right] = 1.$$

So it is clear from the construction that the map $t \mapsto E(t) := \mathbb{E}[f_{\mu,t}(0)]$ is Lipschitz continuous with

$$E(0) = \mathbb{E} \left[\underline{f}_{\mu}(0) \right] \quad \text{and} \quad E(1) = \mathbb{E} \left[\overline{f}_{\mu}(0) \right],$$

which gives us the desired result. \square

The following lemma is similar to Lemma 3.4.

Lemma 3.6. *Assume that $\{\mu_m\}_{m \in \mathbb{N}}$ is a nonnegative sequence converging to μ and f_m is (I_i, V, μ_m) -admissible for each $m \in \mathbb{N}$. Then we have that*

(1) *if $\mu \in \mathcal{P}$, then*

$$\limsup_{m \rightarrow \infty} f_m, \quad \liminf_{m \rightarrow \infty} f_m \quad \text{are} \quad (I_i, V, \mu)\text{-admissible};$$

(2) *if $m_{\min} \geq \overline{m}$ and $\mu \leq m_{\min} - \overline{m}$, then except on a countable set,*

$$\limsup_{m \rightarrow \infty} f_m = \liminf_{m \rightarrow \infty} f_m = \psi_{2L+1}(\mu - V);$$

(3) *if $\mu \geq M_{\max}$, then except on a countable set,*

$$\limsup_{m \rightarrow \infty} f_m = \liminf_{m \rightarrow \infty} f_m = \psi_1(\mu - V).$$

Proof. Denote $f = \limsup_{m \rightarrow \infty} f_m$. The proof for \liminf is similar.

(1) Assume $\mu \in \mathcal{P}$. Let $\{I_i\}_{-\infty < i < \infty}$ be the (μ, V) -admissible decomposition of \mathbb{R} . For fixed $k \in \mathbb{Z}$ and $\varepsilon > 0$, when m is large enough,

$$\begin{cases} \{\mu_m - V(y) : y \in (a_k + \varepsilon, a_{k+1} - \varepsilon)\} \cap \{M_i, m_i \mid 1 \leq i \leq L\} = \emptyset, \\ \{\mu_m - V(y) : y \in (a_{k+1} + \varepsilon, a_{k+2} - \varepsilon)\} \cap \{M_i, m_i \mid 1 \leq i \leq L\} = \emptyset. \end{cases}$$

Hence we can find four natural numbers $1 \leq l, \tilde{l}, q, \tilde{q} \leq 2L+1$ and two subsequences $\{f_{l_n}\}_{n \geq 1}$ and $\{f_{q_n}\}_{n \geq 1}$ such that

$$f|_{I_k} = \psi_l(\mu - V) \quad \text{and} \quad f|_{I_{k+1}} = \psi_q(\mu - V),$$

$$f_{l_n} = \begin{cases} \psi_l(\mu - V) & \text{in } (a_k + \frac{1}{n}, a_{k+1} - \frac{1}{n}) \\ \psi_{\tilde{l}}(\mu - V) & \text{in } (a_{k+1} + \frac{1}{n}, a_{k+2} - \frac{1}{n}) \end{cases}$$

and

$$f_{q_n} = \begin{cases} \psi_{\tilde{q}}(\mu - V) & \text{in } (a_k + \frac{1}{n}, a_{k+1} - \frac{1}{n}) \\ \psi_q(\mu - V) & \text{in } (a_{k+1} + \frac{1}{n}, a_{k+2} - \frac{1}{n}). \end{cases}$$

It suffices to show that for $u \in C^{0,1}(\mathbb{R})$ such that $u' = f$, then u is a solution of (2.4) at a_{k+1} . Consider $u_l \in C^{0,1}(a_k, a_{k+2})$ with

$$u'_l := \begin{cases} \psi_l(\mu - V) & \text{in } I_k \\ \psi_{\tilde{l}}(\mu - V) & \text{in } I_{k+1} \end{cases}$$

and $u_q \in C^{0,1}(a_k, a_{k+2})$ with

$$u'_q := \begin{cases} \psi_{\bar{q}}(\mu - V) & \text{in } I_k \\ \psi_q(\mu - V) & \text{in } I_{k+1} \end{cases}$$

Due to the stability of viscosity solutions, u_l and u_q are both viscosity solutions to (2.4) in (a_k, a_{k+2}) . By using the similar proof as the last part of that of Lemma 3.4, we are done.

(2) Assume $m_{\min} \geq \bar{m}$ and $\mu \leq m_{\min} - \bar{m}$. Note if $u' \geq 0$ and u is a solution of (2.4), we must have that $u' = \psi_{2L+1}(\mu - V)$. So it is clear that (2) holds for $x \in \mathbb{R} \setminus A$ where

$$A := \{y \in \mathbb{R} : V(y) = -\bar{m}\},$$

which is either an empty set or a countable set due to (H4).

(3) Assume $\mu \geq M_{\max}$. Note if $u' \geq 0$ and u is a solution of (2.4), we must have that $u' = \psi_1(\mu - V)$. So it is clear that (3) holds for $y \in \mathbb{R} \setminus B$ where

$$B := \{y \in \mathbb{R} : V(y) = 0\}$$

which is either an empty set or a countable set due to (H4). \square

For each $\mu \geq 0$, define

$$\mathcal{I}_\mu := \begin{cases} [\mathbb{E}[\underline{f}_\mu(0)], \mathbb{E}[\bar{f}_\mu(0)]] & \text{for } \mu \in \mathcal{P}, \\ \{\mathbb{E}[f_\mu(0)]\} & \text{for } \mu \in [0, \infty) \setminus \mathcal{P}. \end{cases}$$

For $\mu \in [0, \infty) \setminus \mathcal{P}$, we also write for consistency that

$$\mathbb{E}[\underline{f}_\mu(0)] = \mathbb{E}[\bar{f}_\mu(0)] = \mathbb{E}[f_\mu(0)].$$

Observe that Lemma 3.5 implies that

$$\begin{aligned} p \in \mathcal{I}_\mu &\Rightarrow \text{existence of sublinear solutions to the cell problem} \\ &\Rightarrow (H, V) \text{ is regularly homogenizable at } p \text{ and } \bar{H}(p) = \mu. \end{aligned}$$

In particular, if \mathcal{I}_μ is not a single point, we obtain a flat piece. These intervals are mutually disjoint:

Lemma 3.7. *If $\mu, \nu \in [0, \infty)$ with $\mu \neq \nu$, then $\mathcal{I}_\mu \cap \mathcal{I}_\nu = \emptyset$.*

Lemma 3.8. *Set $q_0 := \mathbb{E}[\underline{f}_0(0)]$. Then*

$$\bigcup_{\mu \geq 0} \mathcal{I}_\mu = [q_0, \infty).$$

Proof. We divide the proof into two steps.

Step 1. We first show that those intervals \mathcal{I}_μ are upper-semicontinuous with respect to μ , i.e., for any nonnegative sequence $\{\mu_m\}$ converging to μ

$$(3.2) \quad \begin{cases} \mathbb{E}[\bar{f}_\mu(0)] \geq \limsup_{m \rightarrow \infty} \mathbb{E}[\bar{f}_{\mu_m}(0)] \\ \mathbb{E}[\underline{f}_\mu(0)] \leq \liminf_{m \rightarrow \infty} \mathbb{E}[\underline{f}_{\mu_m}(0)] \end{cases}.$$

In fact, owing to Lemma 3.6, it is obvious that

$$\underline{f}_\mu \leq \liminf_{m \rightarrow \infty} \underline{f}_{\mu_m} \leq \limsup_{m \rightarrow \infty} \bar{f}_{\mu_m} \leq \bar{f}_\mu \quad \text{a.e. in } \mathbb{R}.$$

Hence using stationary ergodicity

$$\begin{aligned} \limsup_{m \rightarrow \infty} \mathbb{E} [\bar{f}_{\mu_m}(0)] &= \limsup_{m \rightarrow \infty} \int_0^1 \mathbb{E} [\bar{f}_{\mu_m}(y)] dy \\ &\leq \int_0^1 \mathbb{E} \left[\limsup_{m \rightarrow \infty} \bar{f}_{\mu_m}(y) \right] dy \\ &\leq \int_0^1 \mathbb{E} [\bar{f}_\mu(y)] dy = \mathbb{E} [\bar{f}_\mu(0)]. \end{aligned}$$

Similarly, we can show that

$$\liminf_{m \rightarrow \infty} \mathbb{E} [\underline{f}_{\mu_m}(0)] \geq \mathbb{E} [\underline{f}_\mu(0)].$$

Step 2. This part is similar to the proof of the intermediate value theorem for continuous functions. We argue by contradiction. If the conclusion of this lemma is not true, then there exists $\bar{p} > q_0$ such that $\bar{p} \notin \mathcal{I}_\mu$ for all $\mu \geq 0$.

For $\mu, \tilde{\mu} \geq 0$, if

$$\max\{a : a \in \mathcal{I}_\mu\} < \bar{p} < \min\{a : a \in \mathcal{I}_{\tilde{\mu}}\},$$

then we compare \bar{p} with the endpoints of $\mathcal{I}_{\frac{\mu+\tilde{\mu}}{2}}$. By repeating this procedure, we can find two sequences $\{\mu_n\}_{n \geq 1}$ and $\{\tilde{\mu}_n\}_{n \geq 1}$ such that

$$\lim_{n \rightarrow \infty} \mu_n = \lim_{n \rightarrow \infty} \tilde{\mu}_n = \bar{\mu} \geq 0$$

and for all $n \in \mathbb{N}$

$$\max\{a : a \in \mathcal{I}_{\mu_n}\} < \bar{p} < \min\{a : a \in \mathcal{I}_{\tilde{\mu}_n}\}.$$

Then (3.2) implies that $\bar{p} \in \mathcal{I}_{\bar{\mu}}$, which is a contradiction. \square

We recall now that $\tilde{L} = 0$ and thus H is strictly decreasing on $(-\infty, 0]$. Let

$$\Psi := (H|_{(-\infty, 0]})^{-1} \quad \text{and} \quad q_{-1} := \mathbb{E} [\Psi(-V(0))].$$

Sublinear correctors might not exist when $p \in [q_{-1}, q_0]$. Therefore, we need to build a family of subsolutions which are sufficient to get the homogenization result at the minimum level $\{\bar{H} = 0\}$.

Lemma 3.9. *For any $p \in [q_{-1}, q_0]$ and $\delta > 0$ sufficiently small, there exists a stationary function f such that*

$$p = \mathbb{E} [f(0)]$$

and for any $u \in C^{0,1}(\mathbb{R})$ with $u' = f$, u is a viscosity subsolution of

$$(3.3) \quad H(u') + V(y) = \delta \quad \text{in } \mathbb{R}.$$

Proof. Choose δ such that

$$0 < \delta < \frac{1}{2} \min \{\bar{m}, m_{\min}\},$$

which implies that $\{p : H(p) < \delta\}$ is an interval containing 0.

Take $\{b_i\}_{i \in \mathbb{Z}}$ to be a strictly increasing sequence satisfying $\lim_{i \rightarrow \pm\infty} b_i = \pm\infty$, $V(b_i) = -\delta/4$, and

$$-\frac{\delta}{4} \notin \{V(y) : y \in (b_i, b_{i+1})\}.$$

By (H4) and Lemma A.6, $\{b_i\}$ exists and is unique up to a translation of indices in \mathbb{Z} . For each $i \in \mathbb{Z}$, denote

$$\bar{r}_i := \int_{b_i}^{b_{i+1}} \underline{f}_0(y) dy \quad \text{and} \quad \underline{r}_i := \int_{b_i}^{b_{i+1}} \Psi(-V(y)) dy.$$

For $t \in [0, 1]$ and $i \in \mathbb{Z}$, set $r_i(t) := t\bar{r}_i + (1-t)\underline{r}_i$ and

$$f_{0,t} := \begin{cases} t\underline{f}_0 + (1-t)\Psi(-V) & \text{in } (b_i, b_{i+1}) \text{ if } V((b_i, b_{i+1})) \subset (-\delta/4, 0], \\ f_{r_i(t)}(\underline{f}_0, \Psi(-V), (b_i, b_{i+1})) & \text{in } (b_i, b_{i+1}) \text{ if } V((b_i, b_{i+1})) \subset (-\infty, -\delta/4). \end{cases}$$

Due to the choice of δ , we have that for any $u \in C^{0,1}(\mathbb{R})$ such that $u' = f_{0,t}$, u is a subsolution of (3.3). Repeating the last part of the proof of Lemma 3.5 yields the result. \square

The following assertion holds in all dimensions $d \geq 1$ provided (H1) holds.

Lemma 3.10. *Let v_λ be the unique continuous bounded solution of (2.2) for some given $p \in \mathbb{R}$. Then*

$$\mathbb{P} \left[\liminf_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p) \geq 0 \right] = 1.$$

Theorem 3.11. *Assume $\bar{m} \geq \max_{1 \leq i, j \leq L} (M_i - m_j)$. Then (H, V) is regularly homogenizable and the effective Hamiltonian $\bar{H} : \mathbb{R} \rightarrow [0, \infty)$ is quasi-convex.*

Proof. Due to Lemma 2.3, we may assume (3.1).

When $p \geq \mathbb{E} [\underline{f}_0(0)]$, sublinear solutions to the cell problem

$$(3.4) \quad H(p + v') + V = \bar{H}(p) \quad \text{in } \mathbb{R}$$

exist and are given by Lemma 3.5.

When $\mathbb{E} [\Psi(-V(0))] \leq p \leq \mathbb{E} [\underline{f}_0(0)]$, sublinear solutions to cell problem (3.4) might not exist. However, combining Lemma 3.9 and Lemma 3.10, we have that (H, V) is regularly homogenizable at p and

$$\bar{H}(p) = 0$$

When $p \leq \mathbb{E} [\Psi(-V(0))]$, $\bar{H}(p) \geq 0$ is the unique number given by

$$p = \mathbb{E} [\Psi(\bar{H}(p) - V(0))],$$

and cell problem (3.4) has a sublinear solution $v \in C^{0,1}(\mathbb{R})$ with

$$v' = \Psi(\bar{H}(p) - V) - p \quad \text{in } \mathbb{R}.$$

It is clear that such obtained \overline{H} is quasiconvex: \overline{H} is increasing on $[0, \infty)$ and decreasing on $(-\infty, 0]$. \square

4. HOMOGENIZATION BY INDUCTION

4.1. Induction proof. We present first the proof of Theorem 2.2 by induction.

Proof of Theorem 2.2. We prove by induction on $K := \max\{L, \tilde{L}\}$.

Base case. If $K = 0$, the conclusion follows immediately from Lemma 5.1, which is the one-dimensional case of [3].

Inductive hypothesis. Assume that (H, V) is regularly homogenizable for $K \leq k$ for some given $k \geq 0$. We now argue that (H, V) is regularly homogenizable for $K = k + 1$. Assume that $L \geq \tilde{L}$. In light of Lemma 4.1, it suffices to show that (H^+, V) is regularly homogenizable for

$$H^+(p) := \begin{cases} H(p) & \text{for } p \geq 0, \\ C|p| & \text{for } p \leq 0, \end{cases}$$

for some $C > \|H'\|_{L^\infty(\mathbb{R})}$. There are two cases to be considered.

If $\overline{m} \geq \max_{1 \leq i, j \leq L} (M_i - m_j)$, then the conclusion follows from Theorem 3.11. Otherwise, we use Lemmas 4.2 and 4.3 to reduce H^+ to simpler Hamiltonians and use the inductive hypothesis to achieve the result. \square

4.2. Gluing at the minimum point. For some $C > \|H'\|_{L^\infty(\mathbb{R})}$, define

$$H^+(p) := \begin{cases} H(p) & \text{for } p \geq 0, \\ C|p| & \text{for } p \leq 0, \end{cases}$$

and

$$H^-(p) := \begin{cases} H(p) & \text{for } p \leq 0, \\ C|p| & \text{for } p \geq 0. \end{cases}$$

Lemma 4.1. *If both (H^+, V) and (H^-, V) are regularly homogenizable, then (H, V) is also regularly homogenizable and moreover,*

$$(4.1) \quad \overline{H}(p) = \begin{cases} \overline{H}^+(p) & \text{for } p \geq 0, \\ \overline{H}^-(p) & \text{for } p \leq 0. \end{cases}$$

Note that (4.1) is equivalent to the fact that $\overline{H} = \min\{\overline{H}^+, \overline{H}^-\}$.

Proof. It is enough to consider $p \geq 0$ and show that

$$(4.2) \quad \mathbb{P} \left[\lim_{\lambda \rightarrow 0} |\lambda v_\lambda(0, p) + \overline{H}^+(p)| = 0 \right] = 1,$$

where v_λ is the solution of (2.2). Let v_λ^+ be the solution of

$$(4.3) \quad \lambda v_\lambda^+ + H^+(p + (v_\lambda^+)') + V(y) = 0 \quad \text{in } \mathbb{R}.$$

By the usual comparison principle,

$$(4.4) \quad \|\lambda v_\lambda(\cdot, p)\|_{L^\infty(\mathbb{R})}, \|\lambda v_\lambda^+(\cdot, p)\|_{L^\infty(\mathbb{R})} \leq H(p) + \overline{m},$$

and $v_\lambda \geq v_\lambda^+$ as $H^+ \geq H \geq 0$. Hence

$$(4.5) \quad \mathbb{P} \left[\limsup_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p) \leq \limsup_{\lambda \rightarrow 0} -\lambda v_\lambda^+(0, p) = \overline{H}^+(p) \right] = 1.$$

When $\overline{H}^+(p) = 0$, (4.5) and Lemma 3.10 imply (4.2) immediately. Note also that $\overline{H}^+(0) = 0$. We thus only need to consider the case $p > 0$ and $\overline{H}^+(p) > 0$.

As (H^+, V) is regularly homogenizable,

$$(4.6) \quad \mathbb{P} \left[\forall R > 0, \limsup_{\lambda \rightarrow 0} \max_{|y| \leq R/\lambda} \left| \lambda v_\lambda^+(y, p) + \overline{H}^+(p) \right| = 0 \right] = 1.$$

Fix V belonging to this event. There exists $\lambda(R, V) > 0$ such that for all $\lambda < \lambda(R, V)$ and $y \in B_{R/\lambda}$

$$(4.7) \quad -\lambda v_\lambda^+(y, p) \geq \frac{\overline{H}^+(p)}{2}.$$

We claim that, for $R > 8(H(p) + \overline{m})/p$ and $\lambda < \lambda(R)$,

$$(4.8) \quad p + (v_\lambda^+(y, p))' \geq 0 \quad \text{for a.e. } y \in B_{R/(2\lambda)}.$$

If (4.8) were false, there would exist $y_0 \in B_{R/(2\lambda)}$ such that $p + (v_\lambda^+(y, p))'(y_0) < 0$. On the other hand, by (4.4) and the choice of R ,

$$\frac{2\lambda}{R} \int_{R/(2\lambda)}^{R/\lambda} (p + (v_\lambda^+)'(y)) dy \geq \frac{p}{2} > 0.$$

We use Lemma 2.6 to yield the existence of $y_1 \in (y_0, R/\lambda)$ such that $0 \in D^- u_\lambda^+(y_1)$ for $u_\lambda^+ = py + v_\lambda^+(y, p)$, and hence

$$\lambda v_\lambda^+(y_1, p) + H^+(0) + V(y_1) \geq 0,$$

which contradicts (4.7). Therefore, (4.8) is true and

$$\lambda v_\lambda^+ + H(p + (v_\lambda^+)') + V(y) = 0 \quad \text{in } B_{R/(2\lambda)}.$$

The comparison result in Lemma 2.5 gives

$$\lambda |v_\lambda^+(0, p) - v_\lambda(0, p)| \leq \frac{C}{R}.$$

Sending $R \rightarrow \infty$ to get the conclusion. \square

4.3. Gluing results in case the oscillation of V is not large. We assume that $\tilde{L} = 0$ and

$$(4.9) \quad \overline{m} < \max_{1 \leq i, j \leq L} (M_i - m_j)$$

in Lemma 4.2 and 4.3. Due to (iii) in (H2), there exist unique $k, l \in \{1, \dots, L\}$ such that

$$M_k = M_{\max} \quad \text{and} \quad m_l = m_{\min}.$$

Then of course $\overline{m} < M_k - m_l = \max_{1 \leq i, j \leq L} (M_i - m_j)$. We need to consider two cases $l > k$ and $l \leq k$ as the nature of the difficulties is different.

4.4. Left steep side. We consider first the case that $l > k$.

Let $H_1 : \mathbb{R} \rightarrow [m_l, \infty)$ be a coercive Lipschitz continuous function satisfying that $H_1 \geq H$ and

$$\begin{cases} H_1 = H & \text{on } (-\infty, p_{2k}], \\ H_1 & \text{is strictly increasing in } (p_{2k}, \infty), \end{cases}$$

and $H_2 : \mathbb{R} \rightarrow [m_l, \infty)$ be a coercive Lipschitz continuous function so that $H_2 \geq H$ and

$$\begin{cases} H_2 = H & \text{on } [p_{2l}, \infty), \\ H_2 & \text{is strictly decreasing in } (-\infty, p_{2l}), \end{cases}$$

and $H_3 := \max\{H_1, H_2\}$.

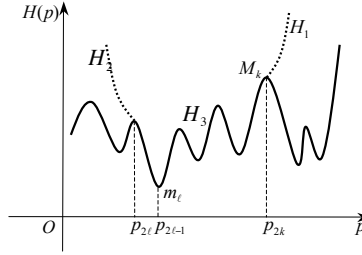


FIGURE 4.1. H_1 , H_2 and H_3 in the gluing lemma 4.2

Lemma 4.2. Assume that (H_i, V) are regularly homogenizable for all $i = 1, 2, 3$. Then (H, V) is also regularly homogenizable and

$$\overline{H} = \min\{\overline{H}_1, \overline{H}_2\}.$$

Proof. It is very easy to verify this lemma in the periodic setting (i.e., V is a periodic function). To highlight the main ideas, we present it here.

Periodic case. It is clear that

$$\overline{H} \leq \min\{\overline{H}_1, \overline{H}_2\}.$$

For fixed $p \in \mathbb{R}$, let $v(\cdot, p) \in C^{0,1}(\mathbb{R})$ be a periodic viscosity solution to the cell problem (3.4). Then we must have either

$$\{p + v'(y, p) : y \in \mathbb{R}\} \subseteq (-\infty, p_{2k}]$$

or

$$\{p + v'(y, p) : y \in \mathbb{R}\} \subseteq [p_{2l}, \infty).$$

Otherwise, due to the periodicity and Lemma 2.6, there exist $y_1, y_2 \in \mathbb{R}$ such that

$$H(p_{2k}) + V(y_1) \leq \overline{H}(p), \quad \text{and} \quad H(p_{2l-1}) + V(y_2) \geq \overline{H}(p).$$

So $\overline{m} \geq M_k - m_l$, which is a contradiction. Hence $v(\cdot, p)$ is either a viscosity to

$$H_1(p + v') + V(y) = \overline{H}(p) \quad \text{in } \mathbb{R}$$

or a viscosity solution to

$$H_2(p + v') + V(y) = \overline{H}(p) \quad \text{in } \mathbb{R}.$$

Accordingly, $\overline{H}(p) = \overline{H}_1(p)$ or $\overline{H}(p) = \overline{H}_2(p)$.

Random case. We note first that $\overline{H}_3(p_{2l-1}) = \min \overline{H}_3 = m_l$. Set

$$A := \{p > p_{2l-1} : m_l < \overline{H}_3(p) < M_k - \overline{m}\}.$$

Step 1. We first show that (2.3) holds for $p \in A$ and

$$(4.10) \quad \overline{H}(p) = \overline{H}_i(p) \quad \text{for } i = 1, 2, 3.$$

The proof of this step is very similar to that of Lemma 4.1 hence is being sketched only. As (H_3, V) is regularly homogenizable, we have

$$\mathbb{P} \left[\forall R > 0, \limsup_{\lambda \rightarrow 0} \max_{|y| \leq R/\lambda} |\lambda v_{3\lambda}(y, p) + \overline{H}_3(p)| = 0 \right] = 1,$$

where $v_{3\lambda}$ is the viscosity solution to

$$(4.11) \quad \lambda v_{3\lambda} + H_3(p + v'_{3\lambda}) + V(y) = 0 \quad \text{in } \mathbb{R}.$$

As usual, $\lambda \|v_{3\lambda}\|_{L^\infty(\mathbb{R})} \leq H_3(p) + \overline{m}$. Set

$$\delta := \min \{ \overline{H}_3(p) - m_l, M_k - \overline{m} - \overline{H}_3(p) \}.$$

There exists $\lambda_3(R, V) > 0$ such that when $\lambda < \lambda_3(R, V)$

$$(4.12) \quad \max_{y \in B_{R/\lambda}} |\lambda v_{3\lambda}(y, p) + \overline{H}_3(p)| \leq \frac{\delta}{8}.$$

Fix $R > 4(H_3(p) + \overline{m})/(p - p_{2l-1})$ and $\lambda < \lambda_3(R, V)$. Then (4.12) yields, for $y \in B_{R/\lambda}$,

$$H_3(p + v'_{3\lambda}(y, p)) \leq \overline{H}_3(p) + \frac{\delta}{8} + \overline{m} < M_k - \frac{\delta}{8}.$$

Therefore, there exists $\tau > 0$ depending only on H, δ such that

$$(4.13) \quad p + v'_{3\lambda}(\cdot, p) < p_{2k} - \tau \quad \text{a.e. in } B_{R/\lambda}.$$

On the other hand, the choice of R allows us to get that

$$\frac{2\lambda}{R} \int_{R/(2\lambda)}^{R/\lambda} (p + v'_{3\lambda}(y, p)) \, dy > p_{2l-1},$$

which yields, by using the same proof as that of (4.8),

$$(4.14) \quad p + v'_{3\lambda}(\cdot, p) > p_{2l-1} \quad \text{a.e. in } B_{R/(2\lambda)}.$$

Combining (4.13) and (4.14) to achieve that

$$(4.15) \quad p_{2l-1} < p + v'_{3\lambda}(\cdot, p) < p_{2k} - \tau \quad \text{a.e. in } B_{R/(2\lambda)},$$

and thus

$$\lambda v_{3\lambda} + H(p + v'_{3\lambda}) + V(y) = 0 \quad \text{in } B_{R/(2\lambda)}.$$

So the comparison result in Lemma 2.5 yields

$$\lambda |v_{3\lambda}(0, p) - v_\lambda(0, p)| \leq \frac{C}{R}.$$

Letting $R \rightarrow \infty$ to conclude Step 1. Since $H_1 = H_2 = H_3$ in $[p_{2l}, p_{2k}]$, (4.15) immediately leads to $\overline{H}_1 = \overline{H}_2 = \overline{H}_3$ in A .

Step 2. We claim that (2.3) holds for $p \leq p_{2l-1}$ and

$$\overline{H}(p) = \overline{H}_1(p).$$

This is due to $\bar{m} < M_k - m_l$. Let $v_{1\lambda}$ be the unique viscosity solution to

$$\lambda v_{1\lambda} + H_1(p + v'_{1\lambda}) + V(y) = 0 \quad \text{in } \mathbb{R}.$$

Then

$$\mathbb{P} \left[\forall R > 0, \limsup_{\lambda \rightarrow 0} \max_{|y| \leq R/\lambda} |\lambda v_{1\lambda}(y, p) + \bar{H}_1(p)| = 0 \right] = 1.$$

Let $\delta_1 := M_k - m_l - \bar{m} > 0$. For each $R > 0$, there exists $\lambda_1(R, V) > 0$ such that, for $\lambda < \lambda_1(R, V)$,

$$\max_{y \in B_{R/\delta}} |\lambda v_{1\lambda}(y, p) + \bar{H}_1(p)| \leq \frac{\delta_1}{8}.$$

Choose $\tau_1 > 0$ sufficiently small so that

$$H_1 < m_l + \frac{\delta_1}{8} \quad \text{in } (p_{2l-1} - \tau_1, p_{2l-1} + \tau_1).$$

For $R > 4(H_1(p) + \bar{m})/\tau_1$ and $\lambda < \lambda_1(R, V)$, we also have the following key property

$$(4.16) \quad p + v'_{1\lambda}(\cdot, p) \leq p_{2k} \quad \text{a.e. in } B_{R/(2\lambda)}.$$

If not, then there exists $y_0 \in B_{R/(2\lambda)}$ such that $p + v'_{1\lambda}(y_0, p) > p_{2k}$. Due to the choice of R ,

$$\frac{2\lambda}{R} \int_{-R/\lambda}^{-R/(2\lambda)} v'_{1\lambda}(y, p) dy < \tau_1, \quad \frac{2\lambda}{R} \int_{R/(2\lambda)}^{R/\lambda} v'_{1\lambda}(y, p) dy < \tau_1.$$

According to Lemma 2.6, there must exist $y_+ \in (y_0, R/\lambda)$ and $y_- \in (-R/\lambda, y_0)$ such that

$$H_1(p_{2l-1} + \tau_1) + V(y_-) \geq -\lambda v_{1\lambda}(y_-, p)$$

and

$$H_1(p_{2k}) + V(y_+) \leq -\lambda v_{1\lambda}(y_+, p).$$

Hence $\bar{m} \geq M_k - m_l - \delta_1/2$, which contradicts the choice of δ_1 . Thus, (4.16) holds and

$$\lambda v_{1\lambda} + H(p + v'_{1\lambda}) + V(y) = 0 \quad \text{in } B_{R/(2\lambda)},$$

and, in light of Lemma 2.5,

$$\lambda |v_{1\lambda}(0, p) - v_\lambda(0, p)| \leq \frac{C}{R}.$$

Step 2 is complete.

Step 3. By similar arguments as in the above two steps, we can conclude that

- For $p \geq p_{2k}$ then (2.3) holds and $\bar{H}(p) = \bar{H}_2(p)$.
- For $p \in \mathbb{R}$ such that $\bar{H}_1(p) < M_k - \bar{m}$, then (2.3) is true and $\bar{H}(p) = \bar{H}_1(p)$.
- For $p \in [p_{2l-1}, p_{2k}]$ with $\bar{H}_2(p) > m_l$, then (2.3) holds and $\bar{H}(p) = \bar{H}_2(p)$.
- For $p \in [p_{2l-1}, p_{2k}]$ and $\bar{H}_2(p) < M_k - \bar{m}$, then $\bar{H}_2(p) = \bar{H}_3(p)$.

Since $H_3 = \max\{H_1, H_2\}$, one gets $\bar{H}_3 \geq \max\{\bar{H}_1, \bar{H}_2\}$. In particular, if $p \in [p_{2l-1}, p_{2k}]$ and $\bar{H}_3(p) \geq M_k - \bar{m}$, then by the last assertion above

$$\bar{H}_3(p) \geq \bar{H}_2(p) \geq M_k - \bar{m} > m_l,$$

and thus

$$(p_{2l-1}, p_{2k}) \subset A \cup \{p : \bar{H}_1(p) < M_k - \bar{m}\} \cup \{p \in [p_{2l-1}, p_{2k}] : \bar{H}_2(p) > m_l\}.$$

The proof is complete. \square

4.5. Right side is steeper. We consider now the case $l \leq k$. We cannot simply copy the method when $l > k$. The subtlety is that the decomposition in the previous case will not lead to a simpler Hamiltonian if $l = 1$. We need to employ the following different approach.

Let $H_1 : \mathbb{R} \rightarrow [0, \infty)$ be a coercive Lipschitz continuous function satisfying that $H_1 \geq H$ and

$$\begin{cases} H_1 = H & \text{on } (-\infty, p_{2k}], \\ H_1 & \text{is strictly increasing in } (p_{2k}, \infty), \end{cases}$$

and $H_2 : \mathbb{R} \rightarrow [m_l, \infty)$ be a coercive Lipschitz continuous function so that $H_2 \geq H$ and

$$\begin{cases} H_2 = H & \text{on } [p_{2k}, \infty), \\ H_2 & \text{is strictly decreasing in } (-\infty, p_{2k}). \end{cases}$$

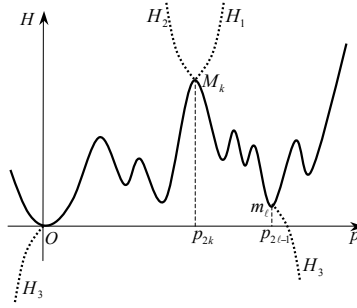


FIGURE 4.2. Graphs of H_1 and H_2 in Lemma 4.3

Lemma 4.3. Assume both (H_1, V) and (H_2, V) are regularly homogenizable. Then (H, V) is regularly homogenizable and

$$\overline{H}(p) = \begin{cases} \overline{H}_1(p) & \text{for } p \leq 0, \\ \min\{\overline{H}_1(p), \overline{H}_2(p), M_k - \overline{m}\} & \text{for } p \in [0, p_{2l-1}], \\ \overline{H}_2(p) & \text{for } p \geq p_{2l-1}. \end{cases}$$

Proof. As in the previous gluing lemma, for readers' convenience, we first prove the second equality in the statement when V is periodic.

Periodic case. For $p \in [0, p_{2l-1}]$, let $v(\cdot, p) \in C^{0,1}(\mathbb{R})$ be a periodic viscosity solution to (3.4). Since

$$\int_0^1 p + v'(y, p) dy = p \in [0, p_{2l-1}],$$

using Lemma 2.6, it is easy to see that $\overline{H}(p) \leq M_k - \overline{m}$ and therefore

$$\overline{H}(p) \leq \min\{\overline{H}_1(p), \overline{H}_2(p), M_k - \overline{m}\}.$$

If $\overline{H}(p) < M_k - \overline{m}$, then we have either

$$\{p + v'(y, p) : y \in \mathbb{R}\} \subseteq (-\infty, p_{2k})$$

or

$$\{p + v'(y, p) : y \in \mathbb{R}\} \subseteq (p_{2k}, \infty).$$

Otherwise, the periodicity and Lemma 2.6 imply the existence of $y_1 \in \mathbb{R}$ such that

$$M_k - \bar{m} \leq H(p_{2k}) + V(y_1) \leq \bar{H}(p),$$

which contradicts our assumption. Hence v is either a viscosity solution to

$$H_1(p + v') + V(y) = \bar{H}(p) \quad \text{in } \mathbb{R}$$

or v is a viscosity solution to

$$H_2(p + v') + V(y) = \bar{H}(p) \quad \text{in } \mathbb{R}.$$

So $\bar{H}(p) = \bar{H}_1(p)$ or $\bar{H}(p) = \bar{H}_2(p)$.

Random case. Proofs of the first and third equalities in the statement are similar to that of Step 2 in the proof of Lemma 4.2. We will prove the equality in the middle. Using similar arguments to Step 1 in the proof of Lemma 4.2, we can deduce that

Claim 1. For $p \in \mathbb{R}$, if $\bar{H}_1(p) < M_k - \bar{m}$, then (H, V) is regularly homogenizable at p and

$$\bar{H}(p) = \bar{H}_1(p).$$

Claim 2. For $p \in \mathbb{R}$, if $\bar{H}_2(p) < M_k - \bar{m}$, then (H, V) is regularly homogenizable at p and

$$\bar{H}(p) = \bar{H}_2(p).$$

It is easy to see that $\bar{H}_1(0) = 0$ and $\bar{H}_2(p_{2l-1}) = m_l$. Also, since

$$\inf_{y \in \mathbb{R}} \{H_1(p_{2k}) + V(y)\} = \inf_{y \in \mathbb{R}} \{H_2(p_{2k}) + V(y)\} = M_k - \bar{m},$$

we have that $\min\{\bar{H}_1(p_{2k}), \bar{H}_2(p_{2k})\} \geq M_k - \bar{m}$. Now denote

$$q_1 := \min\{p \in [0, p_{2k}] : \bar{H}_1(p) = M_k - \bar{m}\} > 0$$

and

$$q_2 := \max\{p \in [p_{2k}, p_{2l-1}] : \bar{H}_2(p) = M_k - \bar{m}\} < p_{2l-1}.$$

Claim 1 and Claim 2 imply that (H, V) is regularly homogenizable for $p \in [0, q_1) \cup (q_2, p_{2l-1}]$ and

$$\bar{H}(p) = \begin{cases} \bar{H}_1(p) & \text{when } p \in [0, q_1) \\ \bar{H}_2(p) & \text{when } p \in (q_2, p_{2l-1}]. \end{cases}$$

Our next goal is to show that (H, V) is regularly homogenizable for $p \in [q_1, q_2]$ and

$$(4.17) \quad \bar{H}|_{[q_1, q_2]} \equiv M_k - \bar{m}.$$

Owing to Claims 1 and 2, and the stability Lemma 2.3, we have that (H, V) are regularly homogenizable at q_1 and q_2 with

$$\bar{H}(q_1) = \bar{H}(q_2) = M_k - \bar{m}.$$

Now choose $H_3 : \mathbb{R} \rightarrow (-\infty, M_k]$ to be Lipschitz continuous function such that $H \geq H_3$, $\lim_{|p| \rightarrow +\infty} H_3(p) = -\infty$ and

$$\begin{cases} H_3 = H & \text{in } [0, p_{2l-1}] \\ H_3 & \text{is strictly increasing on } (-\infty, 0] \\ H_3 & \text{is strictly decreasing on } [p_{2l-1}, \infty). \end{cases}$$

Using similar arguments as Step 1 in the proof of Lemma 4.2, we have that

Claim 3. (H_3, V) is regularly homogenizable at q_1 and q_2 and

$$\overline{H}_3(q_1) = \overline{H}_3(q_2) = M_k - \overline{m}.$$

Let $H_0(p) := -H_3(p_{2k} - p) + M_k$ for $p \in \mathbb{R}$. It is easy to check that w is the viscosity solution to

$$\lambda w + H_0(p + w') - V - \overline{m} = 0 \quad \text{in } \mathbb{R}$$

if and only if $\tilde{w} = -w$ is a viscosity solution to

$$\lambda \tilde{w} + H_3(p_{2k} - p + \tilde{w}') + V + \overline{m} - M_k = 0 \quad \text{in } \mathbb{R}.$$

By applying Lemma 4.4 to $(H_0, -V - \overline{m})$, we deduce that (H_3, V) is regularly homogenizable at $p \in [q_1, q_2]$ and

$$(4.18) \quad \overline{H}_3|_{[q_1, q_2]} \equiv M_k - \overline{m}.$$

Let $v_\lambda(\cdot, p) \in C^{0,1}(\mathbb{R})$ be the unique bounded viscosity solution to (2.2). Since $H \geq H_3$, (4.18) says that for $p \in [q_1, q_2]$

$$(4.19) \quad \mathbb{P} \left[\liminf_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p) \geq M_k - \overline{m} \right] = 1.$$

Now choose $\tilde{H} : \mathbb{R} \rightarrow \mathbb{R}$ to be a Lipschitz continuous function satisfying that $H \leq \tilde{H}$,

$$\tilde{H}(p_{2k}) = M_k, \quad \tilde{H}(0) = 0, \quad \tilde{H}(p_{2l-1}) = m_l$$

and $\tilde{H}|_{(-\infty, 0]}$ is strictly decreasing, $\tilde{H}|_{[0, p_{2k}]}$ is strictly increasing, $\tilde{H}|_{[p_{2k}, p_{2l-1}]}$ is strictly decreasing and $\tilde{H}|_{[p_{2l-1}, \infty)}$ is strictly increasing (see the figure below.)

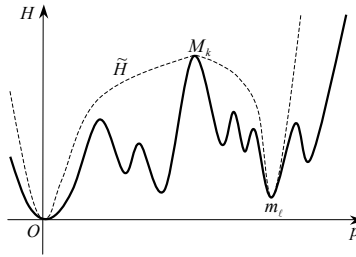


FIGURE 4.3. Graph of \tilde{H}

Since $\overline{m} < M_k - m_l < M_k$, owing to Lemma 5.2, (\tilde{H}, V) is regularly homogenizable and

$$\overline{\tilde{H}}(p) \leq M_k - \overline{m} \quad \text{for } p \in [0, p_{2l-1}].$$

Comparison principle implies that for $p \in [0, p_{2l-1}]$,

$$\mathbb{P} \left[\limsup_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p) \leq M_k - \overline{m} \right] = 1.$$

Combining this with (4.19), we obtain (4.17). \square

Lemma 4.4. *Suppose that (H, V) is regularly homogenizable at $q \in \mathbb{R}$ and $\overline{H}(q) = 0$. Then (H, V) is regularly homogenizable at all points $p \in I$, where I is the line segment between q and 0, and*

$$\overline{H}(p) = 0 \quad \text{for } p \in I.$$

Proof. As usual, we provide first the proof for the periodic case.

Periodic Case. When V is periodic, the proof is quite simple. Assume $q > 0$. It is obvious that $\overline{H}(p) \geq 0$. So we only to verify that $\overline{H}(p) \leq 0$ for $p \in [0, q]$.

Pick $y_0 \in \mathbb{R}$ so that $V(y_0) = 0 = \min_{\mathbb{R}} V$. Let $v(\cdot, q) \in C^{0,1}(\mathbb{R})$ be a viscosity solution to cell problem

$$H(q + v') + V(y) = 0 \quad \text{in } \mathbb{R}$$

subject to the condition that $qy_0 + v(y_0, q) = p - q$. Then

$$(4.20) \quad \lim_{r \rightarrow 0} \|q + v'(\cdot, q)\|_{L^\infty(B_r(y_0))} = 0.$$

For fixed $p \in [0, q]$, set $w(y) := \max\{qy + v(y), 0\}$ in $[y_0, y_0 + 1]$ and extend $w - py$ periodically to \mathbb{R} . Note that (4.20) implies that w is differentiable at y_0 and $w'(y_0) = 0$. Then $h = w - py$ is a periodic viscosity subsolution to

$$H(p + h') + V(y) = 0 \quad \text{in } \mathbb{R}.$$

Thus $\overline{H}(p) \leq 0$.

Random Case. It is enough to consider the case where $q > 0$. Denote

$$M^+ = \max_{1 \leq i \leq L} M_i \quad \text{and} \quad M^- = \max_{1 \leq i \leq \tilde{L}} \widetilde{M}_i.$$

If $\max\{M^+, M^-\} \leq \overline{m}$, (1) follows immediately from Theorem 3.11. Let us consider the case

$$\min\{M^+, M^-\} > \overline{m}.$$

The case that one of them is no larger than \overline{m} is simpler. Write

$$k_+ = \max\{1 \leq i \leq L \mid M_i > \overline{m}\}, \quad k_- = \max\{1 \leq i \leq \tilde{L} \mid \widetilde{M}_i > \overline{m}\}.$$

Let $v_\lambda(\cdot, q)$ be the solution of (2.2) with $p = q$. By the hypothesis,

$$\mathbb{P} \left[\forall R > 0, \lim_{\lambda \rightarrow 0} \sup_{y \in B_{R/\lambda}} |\lambda v_\lambda(y, q)| = 0 \right] = 1$$

and by the ergodic theorem

$$(4.21) \quad \lim_{s \rightarrow \pm\infty} \frac{1}{s} \int_0^s \mathbf{1}_{\{y: -\delta < V(y) \leq 0\}} dy = \mathbb{E} [\mathbf{1}_{\{y: -\delta < V(y) \leq 0\}}] > 0,$$

where

$$\delta = \frac{1}{4} \min \left\{ M_{k_+} - \overline{m}, \widetilde{M}_{k_-} - \overline{m}, \overline{m}, m_{\min} \right\}.$$

There exists $\lambda(R, V) > 0$ such that for $\lambda < \lambda(R, V)$

$$\sup_{y \in B_{R/\lambda}} |\lambda v_\lambda(y, q)| < \delta.$$

In view of (4.21), we can choose a sequence $\{\lambda_n\} \rightarrow 0$ such that for all $n \in \mathbb{N}$, $\lambda_n \in (0, \lambda(\delta, R))$ and

$$\int_{R/(2\lambda_n)}^{R/\lambda_n} \mathbf{1}_{\{y: -\delta < V(y) \leq 0\}} dy, \quad \int_{-R/\lambda_n}^{-R/(2\lambda_n)} \mathbf{1}_{\{y: -\delta < V(y) \leq 0\}} dy > 0.$$

Pick $y_{1n}^+ \in (R/(2\lambda_n), R/\lambda_n)$ and $y_{1n}^- \in (-R/\lambda_n, -R/(2\lambda_n))$ such that $v_\lambda(\cdot, q)$ is differentiable at y_{1n}^\pm and $V(y_{1n}^\pm) \in (-\delta, 0)$. Therefore, $H(q + v'_{\lambda_n}(y_{1n}^\pm)) \leq 2\delta$ and

$$(4.22) \quad q + v'_{\lambda_n}(y_{1n}^\pm) \in (\tilde{p}_{2L}, p_{2L}).$$

On the other hand, for all $y \in B_{R/\lambda_n}$, one has

$$(4.23) \quad H(q + v'_{\lambda_n}(y)) \leq \delta + \overline{m} \leq \min\{M_{k+} - 3\delta, \widetilde{M}_{k-} - 3\delta\}.$$

We combine (4.22), (4.23), and Lemma 2.6 to deduce that

$$(4.24) \quad q + v'_{\lambda_n}(y) \in (\tilde{p}_{2k-}, p_{2k+}) \quad \text{a.e. in } B_{R/(2\lambda_n)}.$$

Let $\hat{H} : \mathbb{R} \rightarrow [0, \infty)$ be a Lipschitz continuous function satisfying that $\hat{H} \geq H$ and

$$\hat{H} = H \quad \text{on } [\tilde{p}_{2k-}, p_{2k+}],$$

$\hat{H}|_{[p_{2k+}, \infty)}$ is strictly increasing and $\hat{H}|_{(-\infty, \tilde{p}_{2k-}]}$ is strictly decreasing. See the following figure.

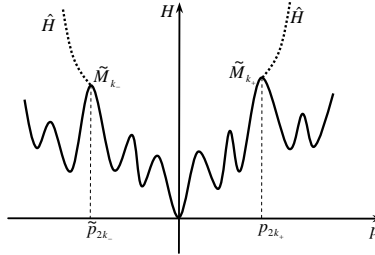


FIGURE 4.4. Graph of \hat{H}

By Theorem 3.11 and Lemma 4.1, (\hat{H}, V) is regularly homogenizable and $\overline{\hat{H}} : \mathbb{R} \rightarrow [0, \infty)$ is quasi-convex. Let \hat{v}_{λ_n} be the solution of

$$(4.25) \quad \lambda_n \hat{v}_{\lambda_n}(y, q) + \hat{H}(q + \hat{v}'_{\lambda_n}) + V(y) = 0 \quad \text{in } \mathbb{R}.$$

Note that both $\hat{v}_{\lambda_n}(\cdot, q)$ and $v_{\lambda_n}(\cdot, q)$ are solutions of (4.25) in $B_{R/(2\lambda_n)}$ by (4.24). We apply Lemma 2.5 to yield

$$\lambda_n |v_{\lambda_n}(0, q) - \hat{v}_{\lambda_n}(0, q)| \leq \frac{C}{R},$$

which of course gives us that $\widehat{H}(q) = 0$. Thus, $\widehat{H} = 0$ on $[0, q]$. Since $\widehat{H} \geq H$, we have that

$$\mathbb{P} \left[\limsup_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p) \leq 0 \right] = 1.$$

Combining with Lemma 3.10, the conclusion follows. \square

5. EXPLICIT FORMULA OF \overline{H} IN CASE THE OSCILLATION OF V IS SMALL

The following lemma is the 1d case of [3]. Since the proof is very easy, we present it here for completeness.

Lemma 5.1. *Suppose that $L = \widetilde{L} = 0$. Then (H, V) is regularly homogenizable and the formula of \overline{H} is given as follows*

$$(5.1) \quad \begin{cases} p = \mathbb{E} [\psi_1(\overline{H}(p) - V(0))] & \text{for } p \geq \mathbb{E} [\psi_1(-V(0))] \\ \overline{H}(p) \equiv 0 & \text{for } p \in [\mathbb{E} [\Psi(-V(0))], \mathbb{E} [\psi_1(-V(0))]] \\ p = \mathbb{E} [\Psi(\overline{H}(p) - V(0))] & \text{for } p \leq \mathbb{E} [\Psi(-V(0))]. \end{cases}$$

Here $\Psi = H^{-1} : [0, \infty) \rightarrow (-\infty, 0]$.

Proof. We only need to prove the middle equality since the other two are obvious due to the existence of sublinear correctors. For $t \in [0, 1]$, denote $u(t) := tu_+ + (1-t)u_-$ where

$$u_+(y) := \int_0^y \psi_1(-V(z)) dz \quad \text{and} \quad u_-(y) := \int_0^y \Psi(-V(z)) dz.$$

Clearly, $u(t)$ is a viscosity subsolution to

$$H(u(t)') + V(y) = 0 \quad \text{in } \mathbb{R}.$$

Moreover, $u(t)' = t\psi_1(-V) + (1-t)\Psi(-V)$ is stationary and $\mathbb{E}[u(t)'(0)] = p(t)$ where

$$p(t) = t\mathbb{E}[\psi_1(-V(0))] + (1-t)\mathbb{E}[\Psi(-V(0))].$$

So we have that

$$\mathbb{P} \left[\limsup_{\lambda \rightarrow 0} -\lambda v_\lambda(0, p(t)) \leq 0 \right] = 1.$$

Combining this with Lemma 3.10 yields the middle equality. \square

Now let us look at the case $\widetilde{L} = 0$ and $L \geq 1$. For convenience, set $m_{L+1} = 0$. We assume in this section that

$$(5.2) \quad \overline{m} < \min_{1 \leq k \leq L} \min\{M_k - m_k, M_k - m_{k+1}\}.$$

We denote, for $1 \leq k \leq L$,

$$\begin{aligned} p_{2k-1}^+ &:= \mathbb{E} [\psi_{2k-1}(m_k - V(0))], & p_{2k-1}^- &:= \mathbb{E} [\psi_{2k}(m_k - V(0))], \\ p_{2k}^+ &:= \mathbb{E} [\psi_{2k}(M_k - \overline{m} - V(0))], & p_{2k}^- &:= \mathbb{E} [\psi_{2k+1}(M_k - \overline{m} - V(0))]. \end{aligned}$$

In light of (5.2), for $1 \leq k \leq L$, $p_{2k} < p_{2k-1}^- < p_{2k-1} < p_{2k-1}^+ < p_{2k-2}$, and $p_{2k+1} < p_{2k}^- < p_{2k} < p_{2k}^+ < p_{2k-1}$.

The following lemma says that (H, V) is regularly homogenizable under assumption (5.2), that is, when the oscillation of V is smaller than the depth of any well in the graph of H .

Lemma 5.2. *We have (H, V) is regularly homogenizable and the formula of \bar{H} is given as follows.*

(1) For $p \in [p_{2k}^+, p_{2k-2}^-]$ where $1 \leq k \leq L$, $\bar{H}(p)$ is given by

$$(5.3) \quad \begin{cases} p = \mathbb{E} [\psi_{2k-1}(\bar{H}(p) - V(0))] & \text{for } p \in [p_{2k-1}^+, p_{2k-2}^-] \\ \bar{H}(p) \equiv m_k & \text{for } p \in [p_{2k-1}^-, p_{2k-1}^+] \\ p = \mathbb{E} [\psi_{2k}(\bar{H}(p) - V(0))] & \text{for } p \in [p_{2k}^+, p_{2k-1}^-]. \end{cases}$$

If $k = 1$, the first equality becomes $p = \mathbb{E} [\psi_1(\bar{H}(p) - V(0))]$ for $p \in [p_1^+, \infty)$.

(2) For $p \in [p_{2k+1}^+, p_{2k-1}^-]$ where $1 \leq k \leq L$, $\bar{H}(p)$ is given by

$$(5.4) \quad \begin{cases} p = \mathbb{E} [\psi_{2k}(\bar{H}(p) - V(0))] & \text{for } p \in [p_{2k}^+, p_{2k-1}^-] \\ \bar{H}(p) \equiv M_k - \bar{m} & \text{for } p \in [p_{2k}^-, p_{2k}^+] \\ p = \mathbb{E} [\psi_{2k+1}(\bar{H}(p) - V(0))] & \text{for } p \in [p_{2k+1}^+, p_{2k}^-]. \end{cases}$$

(3) For $p \leq \mathbb{E} [\psi_{2L+1}(-V(0))]$, $\bar{H}(p)$ is given by

$$\begin{cases} \bar{H}(p) \equiv 0 & \text{for } p \in [\mathbb{E} [\Psi(-V(0))], \mathbb{E} [\psi_{2L+1}(-V(0))]] \\ p = \mathbb{E} [\Psi(\bar{H}(p) - V(0))] & \text{for } p \leq \mathbb{E} [\Psi(-V(0))]. \end{cases}$$

Proof. We only prove (1) as the proofs of (2) and (3) are similar. It suffices to verify the middle equality in (5.3). Other two equalities are obvious due to the existence of sublinear solutions to the cell problem. Our goal is to show that for $p \in [p_{2k-1}^-, p_{2k-1}^+]$

$$(5.5) \quad \mathbb{P} \left[\lim_{\lambda \rightarrow 0} |\lambda v_\lambda(0, p) + m_k| = 0 \right] = 1,$$

where $v_\lambda(\cdot, p) \in C^{0,1}(\mathbb{R})$ is the solution of (2.2).

Let $\tilde{H} \in C^{0,1}(\mathbb{R})$ be a function satisfying that $\tilde{H} \geq H$ (see the figure below), and

$$\begin{cases} \tilde{H} = H & \text{on } [p_{2k}, p_{2k-2}], \\ \tilde{H} & \text{is strictly increasing on } [p_{2k-1}, \infty), \\ \tilde{H} & \text{is strictly decreasing on } (-\infty, p_{2k-1}]. \end{cases}$$

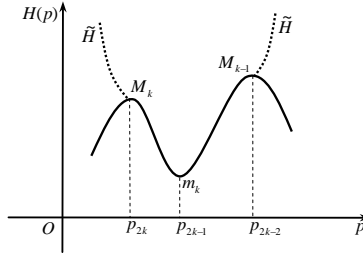


FIGURE 5.1. Construction of \tilde{H}

Owing to the previous lemma, (\tilde{H}, V) is regularly homogenizable and

$$\bar{\tilde{H}}(p) = m_k \quad \text{for } p \in [p_{2k-1}^-, p_{2k-1}^+].$$

Now fix $p \in [p_{2k-1}^-, p_{2k-1}^+]$. Thus, for any $R > 0$,

$$(5.6) \quad \mathbb{P} \left[\limsup_{\lambda \rightarrow 0} \max_{|y| \leq R/\lambda} |\lambda \tilde{v}_\lambda(y, p) + m_k| = 0 \right] = 1,$$

where $\tilde{v}_\lambda(\cdot, p) \in C^{0,1}(\mathbb{R})$ is the unique solution to

$$\lambda \tilde{v}_\lambda + \tilde{H}(p + \tilde{v}'_\lambda) + V(y) = 0 \quad \text{in } \mathbb{R}.$$

It is a routine fact that

$$\sup_{\mathbb{R}} |\lambda v_\lambda(\cdot, p)|, \quad \sup_{\mathbb{R}} |\lambda \tilde{v}_\lambda(\cdot, p)| \leq H(p) + \overline{m}.$$

Due to (5.2) and (5.6), it is clear that, for fixed V and $R > 0$, there exists $\lambda(R, V) > 0$ such that when $\lambda \leq \lambda(R, V)$,

$$p + \tilde{v}'_\lambda(y, p) \in (p_{2k}, p_{2k-2}) \quad \text{for } y \in B_{R/\lambda}.$$

So $\tilde{v}_\lambda(\cdot, p)$ is also a viscosity solution to (2.2) in $B_{R/\lambda}$. Hence according to Lemma 2.5,

$$|\lambda \tilde{v}_\lambda(0, p) - \lambda v_\lambda(0, p)| \leq \frac{C}{R},$$

where $C \geq 1$ depends only on H and \overline{m} . This completes the proof of (5.5). \square

APPENDIX A. AUXILIARY LEMMAS

A.1. Some general results for viscosity solutions in 1-dimensional space.

Lemma A.1. *Assume that $H \in C(\mathbb{R})$ is coercive and $\min_{\mathbb{R}} H = H(0) = 0$. For any $\mu \geq 0$, there exists a Lipschitz continuous viscosity solution u to*

$$H(u') + V(y) = \mu \quad \text{in } \mathbb{R}$$

such that

$$u' \geq 0 \quad \text{for a.e. } y \in \mathbb{R}.$$

Proof. We present the proof in two steps.

Step 1. We first assume that V is periodic with period 1 and $\mu > 0$. Let \overline{H} be the corresponding effective Hamiltonian. It is easy to see that $\overline{H}(0) = 0$. Choose $p_\mu > 0$ such that $\overline{H}(p_\mu) = \mu > 0$. Let $v \in C^{0,1}(\mathbb{R})$ be a periodic viscosity solution to the cell problem

$$H(p_\mu + v') + V(y) = \mu \quad \text{in } \mathbb{R}.$$

We claim that $u = p_\mu y + v$ satisfies that

$$(A.1) \quad u' > 0 \quad \text{for a.e. } y \in \mathbb{R}.$$

Assume not, then there exists $x_1 \in \mathbb{R}$ such that $u'(x_1) \leq 0$. Since

$$p_\mu = \int_{x_1}^{x_1+1} u'(x) dx > 0,$$

there exists $x_2 > x_1$ such that $u'(x_2) > 0$. Due to Lemma 2.6, we may find $x_3 \in [x_1, x_2]$ such that $0 \in D^-u(x_3)$. By definition of viscosity solutions,

$$H(0) + V(x_3) \geq \mu > 0,$$

which is absurd. Thus (A.1) holds.

Step 2: Now for $n \in \mathbb{N}$, let $V_n \in C(\mathbb{R})$ satisfy that

- $V_n(y) = V(y)$ for $|y| \leq n$.
- $V_n(y + 2n) = V_n(y)$ for all $y \in \mathbb{R}$, $\max_{\mathbb{R}} V_n = 0$ and $\max_{\mathbb{R}} |V_n| \leq \sup_{\mathbb{R}} |V|$;

Then owing to Step 1, for $\mu \geq 0$ and $n \in \mathbb{N}$, there exists $u_n \in C^{0,1}(\mathbb{R})$ such that

$$H(u'_n) + V_n(y) = \mu + \frac{1}{n} \quad \text{in } \mathbb{R},$$

and $u'_n > 0$ a.e. in \mathbb{R} .

Due to the coercivity of H and the uniform boundedness of $\{V_n\}$, u_n is equi-Lipschitz continuous in \mathbb{R} . Without loss of generality, we may assume that

$$u_n \rightarrow u \quad \text{locally uniformly in } \mathbb{R}.$$

By usual stability results of viscosity solutions, u satisfies all the requirements of the lemma. \square

Lemma A.2. *Assume that H satisfies (H1)-(H2) and levels set of V have no cluster points. Let $u \in C^{0,1}(\mathbb{R})$ be a viscosity solution of*

$$H(u') + V(y) = \mu \geq 0 \quad \text{in } \mathbb{R}$$

and $u' \geq 0$ a.e. in \mathbb{R} . Then there exists a strictly increasing sequence $\{b_i\}_{i \in \mathbb{Z}}$ such that $\lim_{i \rightarrow \pm\infty} b_i = \pm\infty$ and for $I_i := (b_i, b_{i+1})$, $u \in C^1(I_i)$ and

$$u'|_{I_i} = \psi_{k_i}(\mu - V) \quad \text{for some } k_i \in \{1, 2, \dots, 2L+1\}.$$

Proof. We claim that for any $y \in \mathbb{R}$, there exists $\delta_y > 0$ and $l_y, r_y \in \{1, 2, \dots, 2L+1\}$ such that

$$u' = \begin{cases} \psi_{r_y}(\mu - V) & \text{in } (y, y + \delta_y) \\ \psi_{l_y}(\mu - V(y)) & \text{in } (y - \delta_y, y) \end{cases}$$

Let us prove the first equality. Assume by contradiction that there exist a decreasing sequence $\{y_n\}$ converging to y and two numbers $k, k' \in \{1, 2, \dots, 2L+1\}$ such that $k > k'$, and for all $n \in \mathbb{N}$,

$$(A.2) \quad \begin{cases} u'(y_{2n-1}) = \psi_k(\mu - V(y_{2n-1})) \in [p_k, p_{k-1}], \\ u'(y_{2n}) = \psi_{k'}(\mu - V(y_{2n})) \in [p_{k'}, p_{k'-1}]. \end{cases}$$

This together with Lemma 2.6 yield the existence of a sequence $\{z_n\}$ such that $z_n \in [y_{n+1}, y_n]$ with $p_{k-1} \in D^+u(z_{2n-1})$, and $p_{k-1} \in D^-u(z_{2n})$ for all $n \in \mathbb{N}$. Hence

$$(A.3) \quad H(p_{k-1}) + V(z_{2n-1}) \leq \mu \leq H(p_{k-1}) + V(z_{2n}).$$

By the usual mean value theorem, there exists a further sequence $\{\bar{z}_n\}$ with $\bar{z}_n \in [z_{n+1}, z_n]$ for all $n \in \mathbb{N}$ and

$$(A.4) \quad H(p_{k-1}) + V(\bar{z}_n) = \mu,$$

which implies that y is a cluster point of V , and hence, contradiction. Therefore, (A.2) holds, and furthermore l_y, r_y are unique. Set

$$A = \{y \in \mathbb{R} : l_y \neq r_y\}.$$

By the same reason like the above step, A has no cluster points and we can find a strictly increasing sequence $\{b_i\}_{i \in \mathbb{Z}}$ such that $\lim_{i \rightarrow \pm\infty} b_i = \pm\infty$ and $A \subseteq \{b_i\}_{i \in \mathbb{Z}}$. \square

A.2. Homotopy between solutions. Take $f \in \mathcal{A}(H, V, \mu)$ and $b_1 < b_2 < b_3$ such that for $i = 1, 2$

$$(A.5) \quad f|_{(b_i, b_{i+1})} = \psi_{k_i}(\mu - V) \quad \text{for some } k_i \in \{1, 2, \dots, 2L + 1\}.$$

Denote $k := \min\{k_1, k_2\}$ and

$$\tilde{f} := \begin{cases} f & \text{in } \mathbb{R} \setminus (b_1, b_3), \\ \psi_k(\mu - V) & \text{in } (b_1, b_3). \end{cases}$$

Lemma A.3. *If*

$$(A.6) \quad \{\mu - V(y) : b_1 < y < b_3\} \cap \{M_i, m_j : 1 \leq i, j \leq L\} = \emptyset,$$

then $\tilde{f} \in \mathcal{A}(H, V, \mu)$.

Proof. Assume $k_1 < k_2$. Due to (A.6), both $\psi_{k_1}(\mu - V)$ and $\psi_{k_2}(\mu - V)$ are well defined in (b_1, b_3) . Let

$$q_1 := \psi_{k_1}(\mu - V(b_2)) \quad \text{and} \quad q_2 := \psi_{k_2}(\mu - V(b_2)).$$

Clearly $q_1 > q_2$, $D^+u(b_2) = [q_2, q_1]$, and in light of (A.6) for any $p \in (q_1, q_2)$,

$$H(p) < \mu - V(b_2).$$

We actually can infer furthermore that ψ_{k_1} is strictly increasing, and ψ_{k_2} is strictly decreasing. For any $y \in (b_2, b_3)$, set

$$q_{1,y} := \psi_{k_1}(\mu - V(y)) \quad \text{and} \quad q_{2,y} := \psi_{k_2}(\mu - V(y)).$$

Due to continuity and 2d topology (see the figure below), one still has

$$(A.7) \quad \max_{p \in [q_{2,y}, q_{1,y}]} H(p) = \mu - V(y),$$

which yields that $\tilde{f}_y \in \mathcal{A}(H, V, \mu)$, where

$$\tilde{f}_y = \begin{cases} f & \text{in } \mathbb{R} \setminus (b_1, b_3), \\ \psi_{k_1}(\mu - V) & \text{in } (b_1, y), \\ \psi_{k_2}(\mu - V) & \text{in } (y, b_3). \end{cases}$$

Letting $y \rightarrow b_3$ yields the desired result. The proof for the case $k_1 > k_2$ is similar hence omitted. \square

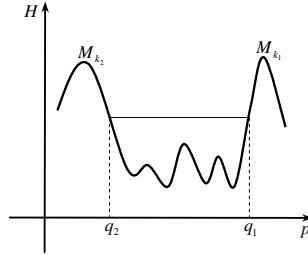


FIGURE A.1. Position of q_1 and q_2

Take $f_1, f_2 \in \mathcal{A}(H, V, \mu)$. Assume there exist $a, b \in \mathbb{R}$ with $a < b$ such that

$$f_1 \geq f_2 \quad \text{in } I := (a, b), \quad f_1 = f_2 \quad \text{on } \mathbb{R} \setminus I.$$

Pick $u_i \in C^{0,1}(\mathbb{R})$ such that $u'_i = f_i$ and $u_i(a) = 0$ for $i = 1, 2$. It is straightforward that

$$u_2(y) \leq u_1(y) \leq u_2(y) - u_2(b) + u_1(b) \quad \text{for } y \in [a, b].$$

For any $c \in [u_2(b), u_1(b)]$ and $y \in I$, denote

$$u_{c,*}(y) := \max\{u_2(y), u_1(y) - u_1(b) + c\}, \quad u_c^*(y) := \min\{u_1(y), u_2(y) - u_2(b) + c\}.$$

Then $u_c^* \geq u_{c,*}$ in I , and u_c^* ($u_{c,*}$) are viscosity supersolution (subsolution) to (2.4) subject to

$$u_c^*(a) = u_{c,*}(a) = 0 \quad \text{and} \quad u_c^*(b) = u_{c,*}(b) = c.$$

For $y \in I$, define

$$u_c(y) := \sup \{w(y) : w \text{ is a subsolution of (2.4) and } u_{c,*} \leq w \leq u_c^* \text{ in } I\}.$$

Also set $f_c = f_c(f_1, f_2, I)$ such that $f_c := u'_c$ in I .

By abuse of notation, we extend f_c to the whole \mathbb{R} as

$$f_c = \begin{cases} u'_c & \text{in } I, \\ f_1 & \text{on } \mathbb{R} \setminus I. \end{cases}$$

Lemma A.4. *For any $c \in [u_2(b), u_1(b)]$, $f_c \in \mathcal{A}(H, V, \mu)$.*

Proof. Let \tilde{u}_c be the extension of u_c to \mathbb{R} as

$$(A.8) \quad \tilde{u}_c := \begin{cases} u_1 & \text{on } (-\infty, a], \\ u_c & \text{in } I, \\ u_1 - u_1(b) + c & \text{on } [b, \infty). \end{cases}$$

We now show that \tilde{u}_c is a viscosity solution of (2.4). It is enough to check the definition of viscosity solutions at $y = a$ and $y = b$. At $y = a$, we have that

$$(A.9) \quad \begin{cases} u_1 \geq \tilde{u}_c \geq u_2 & \text{in } I, \\ u_1 = u_2 = \tilde{u}_c & \text{on } (-\infty, a], \end{cases}$$

and hence $D^-\tilde{u}_c(a) \subset D^-u_1(a)$, $D^+\tilde{u}_c(a) \subset D^+u_2(a)$. These of course imply that \tilde{u}_c is a viscosity solution of (2.4) at $y = a$.

At $y = b$, it is also clear that

$$(A.10) \quad \begin{cases} u_1 - u_1(b) \leq \tilde{u}_c - \tilde{u}_c(b) \leq u_2 - u_2(b) & \text{in } I, \\ u_1 - u_1(b) = u_2 - u_2(b) = \tilde{u}_c - \tilde{u}_c(b) & \text{on } [b, \infty), \end{cases}$$

which gives that \tilde{u}_c is a viscosity solution of (2.4) at $y = b$ by a similar argument like the above. \square

A.3. Approximation of potential V . For $\varepsilon > 0$, consider the approximation of V by analytic functions:

$$V_\varepsilon(y) = \frac{1}{\sqrt{2\pi\varepsilon}} \int_{\mathbb{R}} e^{-\frac{(z-y)^2}{\varepsilon}} V(z) dz.$$

It is easy to check that $V_\varepsilon : \mathbb{R} \rightarrow \mathbb{R}$ is also stationary.

Lemma A.5. *The followings hold*

- (i) $\lim_{\varepsilon \rightarrow 0} \|V_\varepsilon - V\|_{L^\infty(\mathbb{R} \times \Omega)} = 0$.
- (ii) *The level sets of V_ε have no cluster points.*

Proof. The first assertion is obvious. As for (ii), if it were wrong, there would exist $y_0 \in \mathbb{R}$ such that $V_\varepsilon^{(k)}(y_0) = 0$ for all $k \in \mathbb{N}$. Assume without loss of generality that $y_0 = 0$ and $V_\varepsilon(0) = 0$. Then

$$\int_{\mathbb{R}} y^k e^{-\frac{y^2}{\varepsilon}} V(y) dy = 0 \quad \text{for all } k \geq 0.$$

Using Fourier transform, we obtain that $V \equiv 0$, which is absurd. \square

Lemma A.6.

$$\mathbb{P}[\text{for every unbounded interval } I \subset \mathbb{R}, \\ (\inf V, \sup V) \subseteq V(I) := \{V(y) : y \in I\}] = 1.$$

Proof. Using rational numbers, it suffices to show that for any $c \in (\inf V, \sup V) \cap \mathbb{Q}$, $a \in \mathbb{Q}$, $I_a^+ := (a, \infty)$ and $I_a^- := (-\infty, a)$

$$\mathbb{P}[[c, \sup V) \cap V(I_a^+) = \emptyset] = \mathbb{P}[(\inf V, c] \cap V(I_a^+) = \emptyset] = 0$$

and

$$\mathbb{P}[[c, \sup V) \cap V(I_a^-) = \emptyset] = \mathbb{P}[(\inf V, c] \cap V(I_a^-) = \emptyset] = 0.$$

Let $g := \mathbf{1}_{[c, \sup V)}$ and observe, by the ergodic theorem, that

$$\mathbb{P}\left[\lim_{L \rightarrow +\infty} \frac{1}{L-a} \int_a^L g(V(y)) dy = \mathbb{E}(g(V(0))) > 0\right] = 1.$$

This shows that $\mathbb{P}[[c, \sup V) \cap V(I_a^+) = \emptyset] = 0$. The proofs for the other equalities are similar. \square

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